Lecture 8.1

Gravitation

1. Gravitational Force

During our discussion of forces we talked about gravitational force acting on any object near the earth's surface. We have already learned that this force provides the same acceleration of magnitude \( g = 9.8 m/s^2 \) directed to the earth's surface for all the objects near it. The fact that acceleration is the same for all the objects means that this gravitational force is proportional to the mass of the object. This discovery was first made by Isaac Newton during the summer of 1665, when he was watching an apple falling to the ground.

Generalizing this idea, one has to accept that if there is force acting between the earth and any other object, it should be a force of the same nature acting between any objects of nonzero mass. Since there is force acting on the apple from the earth, it should also be force acting from the apple on the earth. According to the Newton's third law, these forces have to have the same magnitude. This means that gravitational force has to be proportional not only to the mass of the first object, \( m_1 \), but also to the mass of the second object, \( m_2 \). In the case of the earth's gravitational force this second mass is the mass of the earth itself. So

\[
F_G \propto m_1 m_2.
\]  

(8.1.1)

Since the mass of any other object near the earth's surface is much smaller than the mass of the earth itself, this is the reason why we can not detect gravitational forces between the other bodies. However, this is not always true. Anybody who lives near the ocean knows about the rise and the fall of the tide. This phenomenon is due to gravitational force, but not the earth's gravitational force. Water rises and falls because of gravitational attraction to the Moon. We also know that this effect occurs on periodic basis, depending on the Moon's position. So, gravitational force depends not only on the masses of interacting objects, but also on their positions. To be precise, it depends on the distance between the interacting objects. The next question to ask is how this force depends on that distance. Newton was the first to work on that question. He made calculations similar to the following example.
Example 8.1.1. What is the Moon’s acceleration due to gravitational force acting on it from the Earth?

Let us notice that gravitational force of the Earth is the only force acting on the Moon (ignoring gravitational forces coming from the other celestial bodies). This force is responsible for the Moon's motion around the Earth, meaning that it provides centripetal acceleration, which is, at the same time, gravitational acceleration, so

\[
a = \frac{v_M^2}{R_M} = \left( \frac{2\pi R_M}{T_M} \right)^2 \frac{1}{R_M} = \frac{4\pi^2 R_M}{T_M^2},
\]

where \( R_M = 3.86 \times 10^8 \text{m} \) is the radius of the Moon's orbit and \( T_M = 27.3 \text{days} = 655 \text{hours} = 2.36 \times 10^6 \text{s} \) is the Moon's rotation period, so

\[
a = \frac{4\pi^2 (3.86 \times 10^8 \text{m})}{(2.36 \times 10^6 \text{s})^2} = 2.73 \times 10^{-3} \text{m/s}^2 = \frac{1}{3590} \text{g}.
\]

This means that gravitational acceleration at the distance \( R_M = 3.86 \times 10^8 \text{m} \) from the Earth's center is 3590 times less than that on the Earth's surface.

Newton noticed that 3590 is about \( 60^2 \) and the distance from the Earth to the Moon is about 60 times larger than the radius of the Earth, so he made an assumption that the force of gravity must be inverse proportional to the square of the distance between the objects. This is how he came up with his law of gravitation, which states the existence of gravitational force between any two particle-like objects and this force has a magnitude of

\[
F = G \frac{m_1 m_2}{r^2},
\]

(8.1.2)

where \( r \) is the distance between the objects and proportionality coefficient \( G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \) is known as gravitational constant. This force is the force of attraction. It acts on each of the interacting particles in the direction along the straight line connecting these particles. This gravitational force is not altered by the presence of other bodies. For all we know, all the particles discovered so far have positive mass, so gravitational force is always the force of attraction.
The value of gravitational constant, \( G \), was determined in the laboratory, using the experiment known as the Cavendish experiment. As you can see this constant is extremely small, so the experiment to determine it must be very sensitive. It can be calculated, if one could measure gravitational force between the two particles, knowing their masses and distance between them, so

\[
G = \frac{Fr^2}{m_1m_2}.
\]

Henry Cavendish in 1797 designed an experiment helping to measure such a small force. He made use of the fact that the force needed to twist a long thin quartz fiber by a few degrees is very small. He had two lead spheres mounted at the ends of the light rod hanging from the fiber. Then he placed two large lead spheres near the smaller spheres and measured the angular deflection of the rod to find the \( G \)-constant.

Example 8.2. Newton did not know about Cavendish’s method, so he estimated gravitational constant based on his estimation of the earth's density \( \rho = 5 \times 10^3 \text{ kg/m}^3 \). What was Newton's estimation for the \( G \)-value?

The gravitational force acting on any object near the earth's surface is \( F=mg \). On the other hand this force can be calculated according to the Gravitational law from the equation 8.2.2, so we have

\[
mg = G\frac{mM_E}{r_E^2},
\]

\[
g = G\frac{M_E}{r_E^2},
\]

\[
G = \frac{g r_E^2}{M_E}.
\]

Here \( M_E \) is mass of the Earth and \( r_E \) is radius of the Earth.

Note that we have considered the earth as a particle for which all the mass is concentrated at its center. It is all right to do so and we will discuss the reasons for that shortly.

Considering earth to be a perfect sphere, its density can be calculated as \( \rho = \frac{M_E}{4\pi \frac{r_E^3}{3}} \), so
\[
G = \frac{G r_E^2}{M_E} = \frac{g r_E^2}{\rho \frac{4\pi}{3} r_E^3} = \frac{3g}{4\pi \rho r_E^3} = \frac{3 \times 9.8 \text{m/s}^2}{4\pi \times 5 \times 10^3 \text{kg/m}^3 \times 6.37 \times 10^6 \text{m}} = 7.35 \times 10^{-11} \text{Nm}^2/\text{kg}^2.
\]

So the Newton's estimation was in the 10% range of the correct value.

As it was sated earlier, the Gravitational law is valid for particles or the objects, separated by the large distances, so that they can be considered as particles. For instance the distance between the Earth and the Moon is large enough compared to their sizes, so one can treat them as particles. This is also true for the apple, which definitely looks like a particle from the earth's perspective. But the earth itself is not a particle-like object compared to the apple, so how can we treat it as such? This question bothered Newton too. Even though he guessed that this was acceptable to consider the interaction of the spherical Earth, with a particle-like object as if they both were particles but he could not prove this at first. So he invented the new area of science, now known as integral calculus. It took him almost 20 years to prove the shell theorem: A uniform spherical shell of matter attracts a particle that is outside of the shell as if all the shell's mass were concentrated at its center.

The important physical principle behind this theorem is known as the principle of superposition. This is the general principle applied to many phenomena, not just gravitational phenomena. It states that the net effect is the sum of individual effects and they do not interfere with each other. For gravity this means, that gravitational force acting on a selected particle is a vector sum of all gravitational forces due to each of the other particles in the system. If the system consists of \(n\) particles, we have the force acting on the first particle to be

\[
\vec{F}_1 = \sum_{i=2}^{n} \vec{F}_{ii} ,
\]

where \(\vec{F}_{ii}\) are the forces acting on the first particle from each of the others \(n-1\) particles. If one considers not a system of the particles, but a large object, such as earth, he/she can consider it as continues system of particles of elementary masses \(dm\), producing elementary gravitational forces \(d\vec{F}\). So the summation in the equation 8.1.3 should be replaced with integration over the object's volume

\[
\vec{F}_1 = \int d\vec{F} .
\]
This is how one can calculate the force acting on the particle near the large object such as earth. In the case of the uniform spherical shell, equation 8.1.4 results in the shell theorem.

**Example 8.1.3** Two spheres of mass $m$ and the third sphere of mass $M$ form a equilateral triangle and the forth sphere of mass $m_4$ is at the center of the triangle. The net gravitational force on that central sphere is zero. What is the mass $M$ in terms of $m$? If we double the value of $m_4$, what then is the magnitude of the net gravitational force on the central sphere?

The picture above shows all four masses and three gravitational forces acting from each of the masses to mass $m_4$. The choice of the coordinate axes is also shown in this picture.

The net force acting on $m_4$ is zero, which means that

$$\sum F_x = 0,$$

$$\sum F_y = 0.$$

Since both masses $m$ are in symmetric positions with respect to mass $m_4$, they both provide the same components of forces in $y$-direction, and the same in magnitude but opposite in sign components in $x$-direction. Moreover, the force acting between $m_4$ and
\( M \) has only \( y \)-component, so the equation for the \( x \)-direction does not involve any unknowns and we do not need it to solve the problem. The only equation necessary is the equation for \( y \)-direction. Let \( r \) be the distance between the central sphere and any of the other three spheres. It is the same for all of them, since this is the equilateral triangle. This means that the gravitational force between the mass \( m_4 \) and any of the masses \( m \) is
\[
F_m = \frac{Gm_4m}{r^2}
\]
and between mass \( m_4 \) and mass \( M \) is
\[
F_M = \frac{Gm_4M}{r^2}.
\]
The equation for the forces' components in \( y \)-direction becomes
\[
F_M - 2F_{my} = 0,
\]
\[
\frac{Gm_4M}{r^2} - 2 \frac{Gm_4m}{r^2} \sin \theta = 0,
\]
\[
M - 2m \sin \theta = 0.
\]
Angle \( \theta \) is shown in the picture. In the case of equilateral triangle \( \theta = 30^\circ \), so
\[
M - 2m \sin 30^\circ = 0,
\]
\[
M = 2m \sin 30^\circ = 2m \frac{1}{2} = m.
\]
So \( M = m \), which was obvious from the beginning, because this is completely symmetric problem. But this is the only reason. You can see from the solution that if the triangle was not equilateral and the angles were not 30 degrees then the answer will be different.

Answering the second question we see, that if one doubles mass \( m_4 \) it will double all three gravitational forces acting on it. Since they all will change in proportional manner this will neither change their directions, nor the fact that they all are of the same magnitude. This means that the net gravitational force will stay zero in this case.

If one assumes the earth to be a uniform sphere of mass \( M_E \) and radius \( r_E \), then, as we saw in the Example 8.1.2, the gravitational acceleration \( g \) near the Earth's surface can be found as
\[
g = \frac{GM_E}{r_E^2}.
\]  \hspace{1cm} (8.1.5)
This provides a method to "weight" the Earth. Since radius of the Earth, gravitational acceleration \( g \) and gravitational constant \( G \) can be measured by other methods, the mass of the Earth will be

\[
M_E = \frac{g r_E^2}{G} = \frac{9.8 \text{ m/s}^2 \left(6.37 \times 10^6 \text{ m}\right)^2}{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2} = 6.0 \times 10^{24} \text{ kg}.
\]

This calculation was done by Cavendish.

However, the Earth is not that perfect as we have assumed and this is why the gravitational acceleration is not the same everywhere, even \( g = 9.8 \text{ m/s}^2 \) is quite an accurate estimation for it.

First the earth is not uniform. Density of the earth varies radially decreasing from the center to the surface. It also varies on the Earth's surface from region to region, which is even more important, since it breaks the spherical symmetry essential for the validity of the shell theorem. Thus gravitational acceleration varies from region to region.

Second the earth is not a perfect sphere. It is an ellipsoid flattened on the poles and bulging at the equator. Its equatorial radius is larger than the polar radius by 21 km. Thus gravitational acceleration increases as one proceeds from the equator towards either pole.

Finally the Earth is rotating, so it is not a perfect inertial reference frame. An object on the Earth's surface anywhere, but at the pole is also rotating together with the Earth and has an additional centripetal acceleration due to this rotation. Indeed if we consider an object at the equator, it has centripetal acceleration of magnitude

\[
a_c = \omega^2 r_E = \left(\frac{2\pi}{T}\right)^2 r_E = \left(\frac{2\pi}{24 \text{ hours}}\right)^2 \left(6.37 \times 10^6 \text{ m}\right) = 0.034 \frac{m}{s^2}.
\]

So the effective free fall acceleration at the equator will be less than gravitational acceleration by 0.034 \( m/s^2 \).

Exercise: Why is the free fall acceleration will be smaller but not larger than the actual acceleration due to gravitational force by the magnitude of the centripetal acceleration?
The most part of the aforementioned corrections affects gravitational acceleration in the second digit after the decimal point, so it is possible to approximate it by $9.8 \text{ m/s}^2$, if one uses only two significant figures.

Newton's shell theorem can also be applied to a particle located inside of a uniform shell. In this case it states, that A uniform shell of matter exerts no net gravitational force on a particle located inside it. This means that the forces coming from different elements of the shell are not disappeared but that the vector sum of all those forces is equal to zero. Considering a particle inside of the earth, gravitational force acting on that particle occurs only due to the portion of the earth with the radius smaller or equal to the particle's distance from the earth's center. The outside part of the earth will not affect this particle. If the earth were a uniform sphere, gravitational force would decrease monotonically closer to the earth's center. However, since the real earth is not uniform it increases first due to the fact that the earth is denser inside, then gravitational force reaches the maximum value at certain depth and only after that tends to zero as the particle approaches the earth's center.

2. Gravitational Energy

Now let us talk about gravitational potential energy. We have only considered the case of the particle near the earth's surface. However, discussing gravitational force in general, we saw that it is not constant but depends on the distance from the earth. Will it be conservative force in that case? Yes, it will. Let us see what gravitational potential energy for this general case is.

We shall consider two particles of masses $m$ and $M$ separated by a distance $r$. In the past we have chosen the reference point for gravitational potential energy to be zero at the earth's surface. It is not a convenient choice, when considering system of two particles which, in general, may have nothing to do with the earth. So now we choose another reference point. We know that gravitational force approaches to zero as separation distance between the particles grows to infinity. It is reasonable to think that the potential energy of the two-particle system is equal to zero, when there is no force acting between them. This means that $r = \infty$ is a good choice of the reference point. To calculate potential energy of this system we will use definition of the potential energy for
the case, where separation distance between the particles is increasing from $R$ to infinity, which gives

$$\Delta U = -\int_R^\infty \tilde{F}(r) \, d\tilde{r} = -\int_R^\infty \left(-\frac{GmM}{r^2}\right) \, dr = -\left[-\frac{GmM}{r}\right]_R^\infty = \frac{GmM}{R}. \quad (8.1.6)$$

Here we have taken into account the fact that the force $\tilde{F}(r)$ and the vector $d\tilde{r}$ have opposite directions. According to our choice of the reference point, equation 8.1.6 gives

$$U(r) = -\frac{GmM}{r}, \quad (8.1.7)$$

which is gravitational potential energy of the system of two particles. We cannot distribute this energy between them, but can only talk about this energy as the energy of the system. If the system consists of more than two particles, one has to consider the algebraic sum of the several terms defined by equation 8.1.7. As for any potential field, the work done by gravitational force does not depend on the shape of the path. Indeed work is only done, when we move a particle of mass $m$ along the radial displacement $r$ from the particle $M$. No work is done for the motion in perpendicular direction, since gravitational force only has component in the radial direction but not in the tangential direction.

**Example 8.1.4.** What is the escape speed of the projectile fired from the earth, so that this projectile will never return back?

When we say that it will never come back, this means that it will go to infinity and stop there. Let us use the energy conservation for the earth-projectile system. We shall call $m$ to be the mass of the projectile, $M_E$ mass of the earth, $r_E$ radius of the Earth and $v$ the original speed of the projectile. The original mechanical energy of the earth-projectile system is

$$E_i = K_i + U_i = \frac{mv^2}{2} - \frac{GmM_E}{r_E}. $$

The final energy of the system is zero, because infinity is the reference point for the gravitational potential energy and the projectile will stop there, so
\[ \frac{mv^2}{2} - \frac{GmM_E}{r_E} = 0, \]
\[ \frac{v^2}{2} - \frac{GM_E}{r_E} = 0, \]
\[ \frac{v^2}{2} = \frac{2GM_E}{r_E}, \]
\[ v = \sqrt{\frac{2GM_E}{r_E}} = \sqrt{\frac{2 \times (6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2) \times (5.98 \times 10^{24} \text{kg})}{6.37 \times 10^6 \text{m}}} = 11.2 \times 10^3 \text{m/s} \]

We saw earlier that one needs a speed of 7.9 km/s to put something on the orbit around the earth. If, however, this speed is larger than 11.2 km/s, this satellite will not stay near the Earth, but it will fly away. As we can see the satellite's speed depends on its orbit, but the total mechanical energy of the satellite-Earth system is always the same. If this satellite moves around the circular orbit of radius \( R \), we have

\[ ma = \frac{GmM_E}{R^2}, \]
\[ m \frac{v^2}{R} = \frac{GmM_E}{R^2}, \]
\[ mv^2 = \frac{GmM_E}{R}, \]
\[ \frac{mv^2}{2} = \frac{GmM_E}{2R}, \]
\[ K = -\frac{U}{2}. \]

Total mechanical energy of this satellite is

\[ E = K + U = -\frac{U}{2} + \frac{U}{2} = -\frac{GmM_E}{R} = -K. \]

So the total energy of the satellite only depends on the radius of its orbit, but not on its speed, which is fixed for a given orbit.

The magnificent example for application of the Newton's law of gravitation is the motion of planets, stars and galaxies, which is governed by that law. The motion of the planets was a subject for scientific research much earlier than Newton discovered his law of gravitation. In fact, people were wandering about planetary motion ever since the ancient time. This motion, being the example of the motion at haven, was considered as a
perfect type of motion. This was the main assumption of the ancient natural philosophy, developed by Aristotle and Ptolemy. Since at that time the Earth was considered to be the center of the universe, everything else considered to be moving around it on a perfect circular orbits. However, watching the motion of planets, for instance Mars, one can see that this is not so. Instead of going around a circle it is producing loops in the sky. This, however, did not stop ancient philosophers from their theory of perfect celestial motion. They introduced not one but many circles going one around the other to describe motion of every planet. For some planets the number of circles was more than a dozen. Such explanation of the "perfect" motion could not satisfy many scientists, resulting in the development of heliocentric system of the world. In this system, not the Earth, but the Sun is the center of the Solar system. All the planets are rotating around it. The Sun itself is also rotating around the center of our galaxy and so on.

To explain complicated motion of planets, three experimental laws were introduced by Johannes Kepler (1571-1630). They are now known under his name. The first Kepler's law, also known as the law of orbits states that

*All planets of the solar system move in elliptical orbits, with the Sun at one focus.*
An ellipse (see the picture) is a closed curve such that the sum of the distances from any point \( P \) on the curve to two fixed points called the foci \( F \) and \( F' \) remains constant. In the picture the planet of mass \( m \) at point \( P \) on the ellipse is moving around the Sun of mass \( M \gg m \) at focus \( F \), so the center of mass of the planet-sun system is approximately at the center of the Sun. The orbit of the planet is described by giving its semimajor axis \( a \) and its eccentricity \( e \). The later is defined in such a way that \( ea \) is the distance from the center of the ellipse to either one of its foci. If \( e = 0 \), the ellipse becomes a circle with both foci merge to its center. For the planets of the solar system the \( e \)-values are quite small, much less than in the picture, so the real planets’ orbits look almost like circles.

Even though one can prove Kepler's first law by solving Newton's equation of motion in gravitational field, it is quite complicated to present here. We can take it just as experimental result.

Kepler's second law or the law of areas states that

“A line that connects a planet to the Sun sweeps out equal areas in the plane of the planet's orbit in equal times, that is the rate \( \frac{dA}{dt} \) at which it sweeps out area \( A \) is constant.”

This tells us that the planet will move slower when it is farther from the Sun and faster when it is closer to the Sun. To prove this law let us take a look at the picture. The velocity vector of the planet \( \vec{v} \) is in the tangential direction to its trajectory. The area \( \Delta A \) swept by the planet’s position vector \( \vec{r} \) for small time \( \Delta t \), when it rotates for a small angle \( \Delta \theta \), is shown by the gray color in the picture. This area can be calculated as

\[
\Delta A = \frac{1}{2} |\vec{r} \times \vec{v}| \Delta t,
\]

since \( |\vec{r} \times \vec{v}| \Delta t \) is the area of the parallelogram formed by vectors \( \vec{r} \) and \( \vec{v} \). So at the limit of infinitely small time change we have

\[
\frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{v}|.
\]

To see that this rate is constant we need to calculate its time derivative, which is

\[
\frac{d}{dt} \left( \frac{dA}{dt} \right) = \frac{d}{dt} \left( \frac{1}{2} |\vec{r} \times \vec{v}| \right) = \frac{1}{2} \left( \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} \right) = \frac{1}{2} \left( |\vec{v} \times \vec{v}| + |\vec{r} \times \vec{a}| \right) = \frac{1}{2} |\vec{r} \times \vec{F}_G| = 0.
\]

At the last step we have made use of the fact that gravitational force \( \vec{F}_G \) between the planet and the Sun acts in the direction opposite to the direction of the vector \( \vec{r} \). We can
also see that Kepler's second law means conservation of the angular momentum for the planet.

The third Kepler's law or the law of periods states that

The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit.

Even though this statement is true for any planet we will only prove it for the case of the circular orbit, which is a special case of the ellipse. Newton's second law in such a case gives us

\[
ma = F_G, \\
mo^2r = \frac{GmM}{r^2},
\]

\[
\left(\frac{2\pi}{T}\right)^2 r = \frac{GM}{r^2},
\]

\[
T^2 = \frac{4\pi^2}{GM} r^3.
\]

During this derivation we have used the fact that centripetal acceleration \(a = \omega^2 r\), where \(\omega\) is the angular speed. It is related to the period \(T\) as \(\omega = \frac{2\pi}{T}\).

The equation holds for any planet of the solar system, provided that we replace the radius by the semimajor axis, which is

\[
\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = const.
\]  
(8.1.8)

So the ratio of the square of the period to the cube of the semimajor axis is the same for all the planets of the same planetary system.

Example 8.1.5. Observations of the light from a certain star indicate that it is part of the binary system. The visible start has orbital speed \(v=270\text{km/s}\), orbital period \(T=1.70\text{days}\) and approximate mass \(m_1 = 6M_S\), where \(M_S = 1.99 \times 10^{30}\text{kg}\) is the Sun's mass. Find the mass of the dark star \(m_2\) if they both are moving on the circular orbits at the same angular speed.
Let us first notice that the masses of those two stars are probably comparable, so we cannot treat them as one star rotating around the other star. Instead we shall consider the center of mass for this system. Since we know from the problem that both stars are moving around the circles, the center of these circles should to be at the system's center of mass. We will only consider the case, when both orbits are at the same plane and since they are rotating at the same angular speed the distance between stars shall always stay the same \( r = r_1 + r_2 \) (see the picture). The only force acting on each of the stars is gravitational force which provides the centripetal acceleration, so the Newton's second law for the first star will be

\[
F_G = m_1 a,
\]

\[
\frac{G m_1 m_2}{r^2} = m_1 \frac{v^2}{r_1},
\]

where \( r_1 \) is the radius of the orbit for the first star, which is the distance from the system's center of mass. Using definition of the center of mass we have

\[
r_1 = \frac{m_1(0) + m_2 r}{m_1 + m_2},
\]

\[
r = r_1 \frac{m_1 + m_2}{m_2}.
\]

The orbital speed of the first star is \( v = \frac{2\pi r_1}{T} \), so \( r_1 = \frac{vT}{2\pi} \). Substituting this back to the Newton's second law we have
The approximate solution of this cubic equation is $m_2 \approx 9 M_S$. 